Technical report associated with the paper “A Semi-Analytical Model for Delay/Doppler Altimetry and its Estimation Algorithm”

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Abstract

The concept of delay/Doppler altimetry has been under study since the mid 90’s, aiming at reducing the measurement noise and increasing the along-track resolution in comparison with the conventional pulse limited altimetry. This paper introduces a new model for the mean backscattered power waveform acquired by a radar altimeter operating in SAR mode, as well as an associated least squares estimation algorithm. As in conventional altimetry, the mean power can be expressed as the convolution of three terms: the flat surface impulse response, the probability density function of the heights of the specular scatterers and the time/frequency point target response of the radar. An important contribution of this paper is to derive an analytical formula for the flat surface impulse response associated with delay/Doppler altimetry. This analytical formula is obtained for circular antenna pattern, no mispointing, no vertical speed effect and a uniform scattering. The double convolution defining the mean echo power can then be computed numerically resulting in a two dimensional semi-analytical model called delay/Doppler map. This delay/Doppler map depends on three altimetric parameters: the epoch, the sea surface wave height and the amplitude. A multi-look model is obtained by summing all the reflected echoes from the same along track surface location of interest after applying appropriate delay compensation (range migration) to align the delay/Doppler map on the same reference. The second contribution of this paper concerns the estimation of the parameters associated with the multi-look semi-analytical model. A least squares approach is investigated by means of the Levenberg-Marquardt algorithm. Simulations conducted on simulated altimetric waveforms allow the performance of the proposed estimation algorithm to be appreciated. The analysis of Cryosat-2 waveforms shows an improvement in parameter estimation when compared to the conventional altimetry.
Index Terms

Altimetry, SAR altimetry, delay/Doppler map, least squares estimation, Cryosat.

I. INTRODUCTION

For more than twenty years, conventional altimeters like Topex, Poseidon-2 or Poseidon-3, have been delivering waveforms which are used to estimate many parameters such as the range between the satellite and the observed scene. The theoretical model for this conventional altimetric waveform is provided by a convolution between three terms that are the flat surface impulse response (FSIR), the probability density function (PDF) of the heights of the specular scatterers and the point target response of the radar (PTR) [1], [2]. Several analytical formulations for the FSIR have been proposed in the literature leading to the Brown model [1] and to more elaborated altimetric waveform models [3]–[5]. Many other studies have been devoted to improve the oceanic analytical model in order to get better estimates of the geophysical parameters. This improvement has been obtained by using different PDF and PTR formulations [2], [6], [7]. Recently, a great effort has also been devoted to process coastal waveforms in order to move the altimetric measurements closer to the coast [8]–[11].

The delay/Doppler altimetry (DDA) proposed in [12] fits into this logic of measurement improvement and has two main objectives. The first one is to reduce the measurement noise by increasing the number of observations (looks) which provide better geophysical parameter estimates. The second one is to increase the along-track resolution which allows the measurements to remain valid until a distance of about 300 meters from the coast (while it was about 10 km for conventional altimetry (CA)). All of these advantages have led to consider DDA in many current and future satellite missions. The first satellite exploiting DDA is the Cryosat-2 satellite which has on board a synthetic aperture interferometric radar altimeter (SIRAL) instrument that includes a DDA mode. Other future missions including DDA are Sentinel-3, Jason-CS (Jason Continuité de Service), and SWOT (Surface Water Ocean Topography), which shows the importance of this new technology.

DDA requires coherent correlation between pulses [12] which is obtained by transmitting pulses with a high pulse repetition frequency (PRF). For instance, the SIRAL altimeter transmits bursts with a frequency of about 85 Hz [13]. Each burst contains 64 coherent pulses (transmitted at a PRF of 18182 Hz) which are processed in order to obtain the delay/Doppler map (DDM)
as shown in Fig. 1. Note that transmitting 64 coherent pulses results in 64 spectral Doppler beams in the DDM, as illustrated in Fig. 1. The exploitation of the DDA oceanic information

![Diagram of delay/Doppler altimeter and construction of delay/Doppler map.]

Fig. 1. Configuration of a delay/Doppler altimeter and construction of a delay/Doppler map.

is based on the analysis of the reflected oceanic waveform called multi-look echo and obtained by applying Doppler processing (slant range correction and multi-looking) to the DDM. This multi-look waveform has a shape that is different from a CA echo, which requires to develop a new altimetric signal model. Many studies have been conducted by different teams for achieving this goal. For instance, numerical models for delay/Doppler (DD) waveforms have been proposed in [14], [15] whereas other models were developed in the SAMOSA project [16], [17].

The first contribution of the present paper is the derivation of a new model for DDA. An analytical model for the FSIR is studied based on a geometrical approach. The proposed FSIR model includes Earth curvature, considers a circular antenna pattern, no mispointing and a Gaussian approximation for the antenna gain as in [1]. The resulting analytical expression of the FSIR is numerically convolved with the PDF of the sea wave height and the PTR of the radar. This yields the mean power of a DDA waveform which depends on three parameters: the epoch $\tau$, the significant wave height $\text{SWH}$ and the amplitude $P_u$.

The second contribution of the paper is to propose and validate an algorithm for estimating
the parameters of the proposed DD semi-analytical model. Many different algorithms have been investigated to estimate the parameters of CA waveforms. These algorithms are for instance based on the maximum likelihood principle [6], [18] or on least squares (LS) techniques [19], [20]. This paper proposes to estimate the geophysical altimetric parameters by a LS approach based on the Levenberg-Marquardt algorithm. The performance of the estimated parameters is analyzed in different scenarios including different noise configurations. Moreover, the evaluation of the estimated parameters, on simulated and real Cryosat-2 data, provides a quantitative measure of the benefits of DDA when compared to CA.

The paper is organized as follows. Section II presents the transition from the conventional altimetric model to the proposed delay/Doppler semi-analytical model. The proposed LS estimation procedure is then introduced in Section III. Section IV validates the proposed model and algorithm with simulated data. The analysis of results associated with real Cryosat-2 waveforms is presented in section V. Conclusions and future work are finally reported in Section VI.
II. Semi-Analytical Model for Delay/Doppler Altimetry

This section first describes the CA model and then introduces the proposed semi-analytical model for DD waveforms. The multi-look processing and the corruption of the waveforms by speckle noise are also described.

A. Conventional Altimetry

In CA, the mean power \( P(t) \) is expressed as the convolution of three terms: the flat surface impulse response (FSIR), the probability density function (PDF) of the heights of the specular scatterers and the point target response of the radar (PTR) as follows

\[
P(t) = \text{FSIR}(t) \ast \text{PDF}(t) \ast \text{PTR}(t)
\]

where \( t \) is the two-way incremental ranging times, i.e., \( t = t' - \frac{2h}{c} \), with \( t' \) the travel time of the echo from the instant of transmission, \( h \) the altitude of the satellite and \( c \) the speed of light. The following subsections describe the three terms of (1).

1) Flat surface impulse response: The FSIR only depends on time and is obtained by integrating over the illuminated area of the surface as follows [1]

\[
\text{FSIR}(t') = \frac{\lambda^2}{(4\pi)^3 L_p} \int_{\mathbb{R}^+ \times [0, 2\pi]} \frac{\delta(t' - \frac{2\pi}{c}) G^2(\rho, \phi) \sigma^0}{r^4} \rho d\rho d\phi
\]

where \( \rho, \phi \) are the radius and the angle of the polar coordinates, \( L_p \) is the two-way propagation loss, \( \lambda \) is the wavelength, \( G \) is the power gain of the radar antenna, \( \delta(t) \) is the delta function, \( \sigma^0 \) is the backscatter coefficient of the surface and \( r = \sqrt{\rho^2 + h^2} \) is the range between the satellite and the observed surface (see Fig. 2). The integral with respect to \( \rho \) in (2) can be expressed in closed form when considering a constant value of \( \sigma^0 \), an antenna without mispointing angles with respect to the \( z \) and \( x \) axes (\( \xi = 0^\circ \) and \( \tilde{\phi} = 0^\circ \) in Fig. 2) and the same gain antenna as in [1], i.e., a Gaussian approximation and a circular antenna pattern. The FSIR is then given by

\[
\text{FSIR}(t) = \frac{P_u}{2\pi} \left(1 + \frac{ct}{2h}\right)^{-3} U(t) \int_0^{2\pi} \exp\left(-\frac{4ct}{\gamma h}\right) d\phi
\]

where \( \gamma = \frac{1}{2} \ln^2 \theta_{3dB}^2 \) is an antenna beam width parameter, \( \theta_{3dB} \) is the half-power antenna beam width, \( P_u = \frac{\lambda^2 G_0^2 \sigma^0}{4(4\pi)^3 L_p h^4} \) is the waveform amplitude, \( G_0 \) is the antenna power gain at boresight and \( U(.) \) denotes the Heaviside function (\( U(t) = 1 \), if \( t \geq 0 \) and \( U(t) = 0 \), if \( t < 0 \)). Equation (3) shows that FSIR\((t)\) is obtained by integrating an appropriate function on a circle
whose radius $\rho(t)$ depends on time, i.e., for each time instant $t$ we have a given radius (see Fig. 3). This radius increases with time since $\rho(t') = \sqrt{\left(\frac{ct}{2}\right)^2 - h^2}$ which reduces to $\rho(t) \simeq \sqrt{hct}$ when considering the approximation $\frac{ct}{h} << 1$ (valid for spaceborne altimetry [1]). Note also, that in CA, we integrate all along the circle of radius $\rho$ (since $\phi \in [0, 2\pi]$) without having a distinction between across-track and along-track directions (axes $x$ and $y$ of Fig. 3 respectively).

The conventional FSIR is finally given by [1], [5]

$$\text{FSIR}(t) = P_u \exp \left(-\frac{4ct}{\gamma h}\right) U(t).$$

where $\left(1 + \frac{ct}{2h}\right)^{-3}$ has been approximated by 1 as in [1] (since $\frac{ct}{h} << 1$).

2) Probability density function of the heights of the specular scatterers: The PDF of the specular points is generally approximated by a Gaussian density whose standard deviation is related to the average SWH [1], [5]

$$\text{PDF}(t) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp \left(-\frac{t^2}{2\sigma_s^2}\right)$$

with $\sigma_s = \frac{\text{SWH}}{2c}$. 

Fig. 2. Geometry used for computing the flat surface impulse response.
Fig. 3. Circles of propagation and Doppler beams. In CA, the FSIR is obtained by integrating over the propagation circles. In DDA, the FSIR is obtained by integrating the energy in the intersection between the propagation circles and the Doppler beams.

3) Radar system point target response: The radar point target response is generally expressed as a squared cardinal sine as follows [5]

\[
\text{PTR}_T(t) = \left| \sin\left(\frac{\pi t}{T}\right) \right|^2
\]

(6)

where \( T = 1/B \) is the sampling period and \( B \) is the reception bandwidth of the altimeter.

B. Delay/Doppler altimetry

As in CA, the mean power \( P(t, f) \) of a DD echo can be expressed as the convolution of three terms: the FSIR, the PDF and the time/frequency PTR [21], [22]. However, unlike the signal \( P(t) \) of (1), the obtained signal \( P(t, f) \) depends on time and Doppler frequency as follows

\[
P(t, f) = \text{FSIR}(t, f) * \text{PDF}(t) * \text{PTR}(t, f)
\]

(7)

where \( f \) denotes the Doppler frequency. The PDF is the same as in (5) and the two other terms are introduced below.
1) Flat surface impulse response: The DDA is pulse-limited across-track and beam-limited along-track as first observed by Raney in [12]. It was proposed in order to increase the along-track resolution by considering the Doppler effect resulting from the satellite velocity. Indeed, the $n$th Doppler frequency $f_n$ is expressed by

$$f_n = \frac{2 \cdot \overrightarrow{r} \cdot \overrightarrow{v_s}}{\lambda |\overrightarrow{r}|} = \frac{2v_s}{\lambda} \cos (\theta_n) \quad (8)$$

where $\overrightarrow{v_s}$ is the satellite velocity and $\theta_n$ is shown in Fig. 4. This figure also shows that

$$\cos (\theta_n) = \frac{y_n(t)}{r_n(t)} = \frac{y_n(t)}{\sqrt{\rho^2(t) + h^2}}, \quad \text{for } t \geq 0 \quad (9)$$

where $y_n(t)$ represents the coordinate of the $n$th along-track beam. Combining (8) and (9) leads to the following expression of $y_n(t)$ as a function of $t$ and $f_n$

$$y_n(t) = \left( \frac{\lambda f_n}{2v_s} \right) \sqrt{\rho^2(t) + h^2}. \quad (10)$$

This equation clearly shows how the coordinate of the along-track beam depends on time. An approximation of (10) is obtained by considering $\rho(t) \ll h$ which is a valid assumption for
near-vertical small angle geometry as explained in [12] (see Appendix B for more details about this approximation). The simplified width of the Doppler beam is then given by [12]

$$y_n = \frac{h\lambda}{2v_s} f_n$$  \hspace{1cm} (11)

with $f_n = (n - 32N_f - 0.5) \frac{F}{N_f}$, for $n \in 1, \cdots, 64N_f$ where $F$ is the frequency resolution obtained from the burst length $\tau_b = 1/F$ (see Fig. 1) and $N_f$ is the frequency oversampling factor. This equation shows that the along-track direction (axis $y$) can be divided into rectangular beams corresponding to different Doppler frequencies displayed in Fig. 3.

Fig. 3 also shows that the computation of the FSIR for DDA is obtained by integrating $\phi$ into rectangular beams defined by fixed coordinates $y_n$ and $y_{n+1}$ (we will consider the time independent Doppler coordinate given in (11) in the rest of the paper). Straightforward computations show that the angles associated with $y_n$ and $y_{n+1}$ are defined by

$$\phi_{t,n} = \text{Re} \left[ \tan \left( \frac{y_n}{\sqrt{\rho^2(t) - y_n^2}} \right) \right]$$

and

$$\phi_{t,n+1} = \text{Re} \left[ \tan \left( \frac{y_{n+1}}{\sqrt{\rho^2(t) - y_{n+1}^2}} \right) \right]$$

(12)

where $\tan(.)$ is the inverse tangent function and $\text{Re}(x)$ denotes the real part of the complex number $x$. As a consequence, the DDA FSIR can be written

$$FSIR(t, n) = \frac{P_u}{2\pi} U(t) \int_{D_{t,n}} \exp \left( \frac{-4ct}{\gamma h} \right) d\phi$$

(13)

where $D_{t,n} = [\phi_{t,n}, \phi_{t,n+1}] \cup [\pi - \phi_{t,n+1}, \pi - \phi_{t,n}]$. Note that the conventional FSIR can be obtained by considering the angles $\phi_{t,n+1} = \frac{\pi}{2}$ and $\phi_{t,n} = -\frac{\pi}{2}$ in $D_{t,n}$. This means that the conventional FSIR given in (4) can also be obtained by summing the signals of all the Doppler beams before range migration, i.e., by summing the DDM rows of Fig. 7.b. As a consequence, (13) leads to following analytical expression of the FSIR

$$FSIR(t, n) = \frac{P_u}{\pi} \exp \left( \frac{-4ct}{\gamma h} \right) (\phi_{t,n+1} - \phi_{t,n}) U(t)$$

(14)

for $n = 1, \cdots, 64N_f$. Note that one has to divide the time $t$ in (4) and (14) by the curvature factor $\alpha = 1 + \frac{h}{R} = 1.11$, where $R = 6378137$ m is the Earth radius, to account for the Earth curvature (see [24], [25] and Appendices C and D for more details about Earth curvature).

\(^1\)A related approach assuming a rectangular shape for the compressed pulse and a rectangular antenna pattern was investigated in [23].
Remark: Wingham et al show in [26] that the Doppler model presents a dependence on height that goes as $h^{-5/2}$ for long delays ($t \to \infty$) and as $h^{-3}$ for small delays (similar to pulse limited altimetry). Considering the first limit ($t \to \infty$) in (14) gives

$$\text{FSIR}(t,n) \sim \frac{\lambda^3 G_0^2 \sigma^0}{128 \pi^3 v_s L_p h^{5/2}} \sqrt{\frac{c}{t}} \exp \left[ -\frac{4ct}{\gamma h} \right] (f_{n+1} - f_n)$$

(15)

when $t \to \infty$, which shows the $h^{-5/2}$ dependence of the leading coefficient. Furthermore, after considering the second limit ($t \to 0^+$), we obtain $\phi_{n+1} = \frac{\pi}{2}$ and $\phi_n = -\frac{\pi}{2}$ because the propagation circle falls entirely within the Doppler beam (as stated in [26]). Then we have

$$\text{FSIR}(t,n) \sim \frac{\lambda^2 G_0^2 \sigma^0}{64 \pi^2 L_p h^3}$$

(16)

when $t \to 0^+$, which shows the $h^{-3}$ dependence of the leading coefficient and its independence on time. These results are in agreement with [12], [26].

2) Radar system point target response: The radar system PTR is composed of temporal and Doppler frequency dimensions. In this paper, we assume that $\text{PTR}(t,f)$ is the multiplication between a temporal function $\text{PTR}_T(t)$ (corresponding to the radar point target response) and a frequency function $\text{PTR}_F(f)$ (resulting from the Doppler processing). This assumption can be justified by recent results available in the literature [21], [22], [27] or by a comparison with the measured Cryosat-2 PTR. Indeed, the actual PTR of the Cryosat-2 altimeter can be estimated by using the instrument calibration data which are obtained by emitting burst of impulses and receiving them inside the system (by skipping the antenna). The temporal component $\text{PTR}_T(t)$ can then be obtained as follows. A range FFT is computed for the received 64 (complex I and Q) signals. The modulus of the resulting signals is then computed. This procedure provides the $\text{PTR}_T(t)$ for each emitted pulse. The 2D PTR associated with delay/Doppler altimetry can also be obtained just by introducing the FFT along-track bloc before the range FFT. The temporal PTR was provided in (6) whereas $\text{PTR}_F(f)$ can be approximated accurately by the following squared sine cardinal function

$$\text{PTR}_F(f) = \left| \frac{\sin \left( \frac{\pi f}{F} \right)}{\pi \frac{f}{F}} \right|^2.$$  

(17)

The resulting PTR is then given by

$$\text{PTR}(t,f) = \text{PTR}_T(t) \text{PTR}_F(f)$$

(18)
Fig. 5 (a) shows the measured Cryosat-2 PTR which is in good agreement with the proposed theoretical PTR shown in Fig. 5 (b). Figs. 6 (a) and (b) compare the measured PTR\(_T\) (PTR\((t,f)\) evaluated at the central beam) and PTR\(_F\) (PTR\((t,f)\) evaluated at the central time) with the proposed square cardinal sines PTR\(_T\) and PTR\(_F\). These figures confirm the good agreement between the measured and theoretical PTRs. It is interesting to note that another PTR could be used without modifying significantly the proposed approach (e.g., PTR\((t,f)\) might be obtained from real measurements). Indeed, the PTR will be convolved numerically with the analytical FSIR derived in this paper and the PDF defined in (5).

![Fig. 5. 2D Radar system point target response. (a) Actual Cryosat-2 PTR\((t,f)\). (b) theoretical PTR\((t,f)\).](image)

### C. Reflected power

The reflected DDA power \(P(t,f)\) (resp. \(P(t)\) for CA) is obtained by a numerical computation of the double convolution (7) (resp. (1)) between the analytical expressions (14), (5) and (18) (resp. (4), (5) and (6)). This convolution is conducted after oversampling the analytical functions in order to better represent the cardinal sines. Appropriate temporal and frequency oversampling factors have been determined by cross-validation yielding \(N_t = 16\) and \(N_f = 15\). The resulting oversampled signal is finally undersampled to obtain the required \(64 \times 128\) DDM. The proposed model (7) is semi-analytical in the sense that an analytical formulation is proposed for the FSIR but that the double convolution in (7) is computed numerically. Note that the proposed semi-analytical model might be modified by introducing a measured PTR\((t,f)\) and/or a PDF different from (5).
D. Multi-looking

Section II-B derived an analytical model for the FSIR\((t, f)\) which is convolved by PDF\((t)\) and PTR\((t, f)\) to compute the reflected power \(P(t, f)\). We also showed previously that each time instant \(t\) is related to a circle of radius \(\rho(t)\) while each Doppler frequency is related to a rectangular along-track beam. Fig. 7 summarizes the construction of a DDM. The signal of a given beam is obtained by summing the energies of all scatterers belonging to this beam. For instance, the energy of the signal corresponding to time instant \(\text{“}k\text{”}\) and Doppler beam \(\text{“}n\text{”}\) is obtained by summing the energies of all scatterers belonging to the intersection of the circle of radius \(\rho(k)\) with the rectangular nadir beam \(\text{“}n\text{”}\). Note that the rises of the reflected powers in the different Doppler beams occur at different time instants (according to Fig. 7 the rise occurs at time instant \(k\) for the nadir beam, at time instant \(3k\) for beams \(\text{“}n+i\text{”}\) and \(\text{“}n-i\text{”}\), etc.). This time shift is related to the range between the satellite and each Doppler beam. Fig. 7.b shows an example of DDM obtained by the proposed model. The parabolic shape of this waveform results from the time shifts between the different beams. The multi-looking process aims at gathering all the reflected energies from a single beam. For that, we first have to compensate the time differences between the different beams in order to have signals rising at the same time instant \(k\). This procedure is called delay compensation [12] or range migration. The delay of each beam \(\delta r_{n}\) is obtained by the difference between the modulus of the position vector \(r_{n} = \sqrt{h^2 + y_{n}^2}\)
(range between the satellite and the Doppler beam \( n \)) and the minimum satellite-surface distance \( h \) [12]

\[
\delta r_n = r_n - h = \sqrt{h^2 + y_n^2} - h. \tag{19}
\]

Note that (19) can be simplified (as proposed in [12]) by considering \( y_n \ll h \) as follows

\[
\delta r_n = h \sqrt{1 + \left( \frac{y_n}{h} \right)^2} - h \approx \frac{y_n^2}{2h} = \frac{h \lambda^2}{8v_s^2 f_n^2}. \tag{20}
\]

Note also that the Earth curvature can be considered by introducing a factor \( \alpha \) yielding [12]

\[
\delta r_n = \sqrt{h^2 + \alpha y_n^2} - h \approx \frac{h \lambda^2}{8v_s^2 f_n^2}. \tag{21}
\]

After delay compensation, the signals associated with the Doppler beams are summed to obtain the multi-look waveform as follows [22] (see Fig. 8).

\[
s(t) = \sum_{n=1}^{64} P (t - \delta t_n, f_n) \tag{22}
\]

where \( \delta t_n = \frac{2 \delta r_n}{c} \) is the delay compensation expressed in seconds. This result is obtained by assuming that each ground Doppler beam is observed by 64 different Doppler beams where
Fig. 8. Delay/Doppler map after delay compensation (left), migrated signals for all Doppler beams (middle) and the corresponding multi-look waveform (right).

Each of them results from averaging $L$ observations (see also the next section). Note that the procedure is quite different for real waveforms where we have to collect the reflected energies of different bursts. For example, the selected scene’s beam may reflect energy coming from nadir beam (beam #33) for the burst $i_1$, from the beam #34 for the burst $i_2$, etc. Note that this stacking procedure aims at reducing the noise effect and that it assumes that the geophysical parameters of the selected beam do not change from one burst to another.

The signal $s(t)$ is finally sampled at instants $t_k = (k - N_t \tau) \frac{T}{N_t}$, for $k = 1, \cdots, KN_t$, where $\tau$ is the epoch and $K = 104$ is the number of samples (without oversampling). An example of resulting DDA vector $s = (s_1, \cdots, s_K)^T = [s(t_1), \cdots, s(t_K)]^T$ is shown in Fig. 9 and compared with the CA echo. The DD echo has a peaky shape around the epoch $\tau$ because of delay compensation. This peaky shape was first quantified in [28] as characteristic of a beam-limited altimeter.
Fig. 9. Delay/Doppler and conventional echoes for the same altimetric parameters ($P_u = 1, \tau = 31$ gates, SWH = 2 m).

E. Speckle noise

In order to generate realistic data similar to Cryosat-2 echoes, the DDM has to be corrupted by speckle noise. Following the works of [26], a multiplicative speckle noise is applied to the DDM leading to

$$y(t) = \sum_{n=1}^{64} P(t - \delta t_n, f_n) q(t - \delta t_n, n)$$

(23)

where $q(t - \delta t_n, n)$ is a random variable distributed according to a gamma distribution $G(L, 1/L)$ (see [31, p. 87] for the definition of the gamma distribution) and $L$ is the number of bursts observing each Doppler beam ($L = 4$ in our simulations).

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$^2$In [26], a single-look is assumed to follow an exponential distribution. Moreover, and as mentioned in [29], [30], each Doppler beam is observed by $L$ bursts (denoted as $N_{\text{obs}}$ in [29], [30]). Thus, the signal of each beam results from the averaging of $L$ observations. It results that the noise corrupting each beam follows a gamma distribution $G(L, 1/L)$. 
III. PARAMETER ESTIMATION

A. Estimation algorithm

This paper proposes to estimate the parameters of the multi-look waveform by using a LS procedure (as for CA [19], [20]). The altimetric waveform \( y = (y_1, \ldots, y_K)^T \) is a noisy version of \( s = (s_1, \ldots, s_K)^T \) which depends on the parameter vector \( \theta = (\theta_1, \theta_2, \theta_3)^T = (\text{SWH}, P_u, \tau)^T \) (the estimation is done under the assumption that \( \xi = 0^\circ \) in both conventional and delay/Doppler models). The LS method consists of estimating the unknown parameter vector \( \theta \) as follows

\[
\arg\min_{\theta} G(\theta) = \arg\min_{\theta} \frac{1}{2} \sum_{k=1}^{K} g_k^2(\theta)
\]  

(24)

where \( g_k(\theta) = y_k - s_k(\theta) \) is a vector of residues. Since \( g_k(\theta) \) is a complicated nonlinear function of SWH and \( \tau \), the optimization problem (24) does not admit a closed-form expression. In this paper, we propose to solve (24) using a numerical optimization method based on the Levenberg-Marquardt algorithm [32]. The parameter update of the iterative Levenberg-Marquardt algorithm is defined by

\[
\theta^{(i+1)} = \theta^{(i)} + e^{(i)}
\]

where \( \theta^{(i)} \) is the estimate of \( \theta \) at the \( i \)th iteration. The choice of \( e^{(i)} \) is based on a Taylor expansion (at the first order) of \( g \) in the neighborhood of \( \theta^{(i)} \)

\[
g(\theta^{(i)} + e^{(i)}) \simeq l(e^{(i)}) = g(\theta^{(i)}) + J(\theta^{(i)}) e^{(i)}
\]  

(25)

where \( J(\theta) = [J_1(\theta), J_2(\theta), J_3(\theta)] = \left[ \frac{\partial g(\theta)}{\partial \theta_1}, \frac{\partial g(\theta)}{\partial \theta_2}, \frac{\partial g(\theta)}{\partial \theta_3} \right] \) is a \( K \times 3 \) matrix representing the gradient of \( g \). After replacing (25) in (24) (and removing notation \( (i) \) for brevity), the following result is obtained

\[
G(\theta + e) \simeq L(e) = \frac{1}{2} l(e)^T l(e) = G(\theta) + e^T J(\theta)^T g + \frac{1}{2} e^T J(\theta)^T J(\theta) e.
\]  

(26)

The descent direction \( e \) is then obtained by minimizing \( L(e) \). By setting to 0 the derivative

\[
L'(e) = J(\theta)^T g + J(\theta)^T J(\theta) e,
\]

we obtain

\[
J(\theta)^T J(\theta) e = -J(\theta)^T g.
\]  

(27)

This relation is the basis of the Gauss-Newton recursion [32], [33]. Levenberg and Marquardt proposed to add a regularization parameter \( \mu \) in (27) leading to

\[
[J(\theta)^T J(\theta) + \mu I_3] e = -J(\theta)^T g
\]  

(28)
where $I_3$ is the $3 \times 3$ identity matrix. Note that the parameter $\mu$ controls the convergence speed of the algorithm. Note also that the derivatives appearing in $J(\theta)$ can be computed numerically by finite difference as follows

$$J_i(\theta) = -\frac{\partial s(\theta)}{\partial \theta_i} \approx -\frac{s(\theta_i + \Delta \theta_i) - s(\theta_i)}{\Delta \theta_i}$$

(29)

with $\Delta \theta = (\Delta \text{SWH}, \Delta \tau, \Delta P_u)^T$. In our simulations, we have chosen $\Delta \theta = (0.05 \text{ m}, 0.02 \text{ gates}, 0.05)^T$.

B. Estimation performance

This section introduces the criteria used to evaluate the quality of the estimators resulting from the proposed model. The quality of the estimation for simulated waveforms can be measured by comparing the estimated and true parameters by using the root mean square error (RMSE)

$$\text{RMSE}(\theta_i) = \sqrt{\frac{1}{m} \sum_{\ell=1}^{m} \left[ \theta_i - \hat{\theta}_i(\ell) \right]^2}, \ i = 1, \cdots, 3$$

(30)

where $\theta_i$ is the true parameter, $\hat{\theta}_i(\ell)$ is the estimated parameter for the $\ell$th waveform and $m$ is the number of simulated waveforms. The bias and the standard-deviation (STD) of the estimator given by

$$\text{Bias}(\theta_i) = \frac{1}{m} \sum_{\ell=1}^{m} (\hat{\theta}_i(\ell) - \theta_i) = \bar{\theta}_i - \theta_i$$

(31)

and

$$\text{STD}(\theta_i) = \sqrt{\frac{1}{m} \sum_{\ell=1}^{m} \left[ \hat{\theta}_i(\ell) - \bar{\theta}_i \right]^2}$$

(32)

can also be used to better analyze the obtained results. The normalized reconstruction error (NRE) given by

$$\text{NRE} = \sqrt{\frac{\sum_{k=1}^{K} (y_k - s_k)^2}{\sum_{k=1}^{K} y_k^2}}$$

(33)

has been computed in order to evaluate the performance in the case of a real waveform. In this case, the estimated delay/Doppler parameters will be compared to the estimated CA parameters.
IV. RESULTS FOR SIMULATED DATA

This section first describes how simulated echoes have been generated. The behavior of the proposed DD model as a function of the Doppler frequency is then analyzed. The effect of range migration on the performance of the LS estimator is finally investigated. The estimation performance of DDA and CA are also compared in order to illustrate the expected improvement of the DD mode (as shown in [14], [34] for a simulated scene and in [21] for another Doppler model).

A. Simulation scenario

This section describes how Cryosat-2 echoes have been generated and introduces the denominations of the related simulated echoes. The Cryosat-2 altimeter called SIRAL presents three modes that are: the low resolution mode (LRM), the synthetic aperture radar mode (SARM) and the synthetic aperture radar interferometric mode (SARInM) [13]. The data of the LRM are used to generate CA echoes (also denoted by CA-LRM echoes) while those of SARM provide DD echoes. However, as the two modes operate separately, the collected data do not result from the same scene and can not be used to compare the same scenario. Hence, the data of SARM are also used to generate conventional echoes called in the present paper CA-SARM for conventional altimetric echoes from SAR mode³. However, the resulting echoes are affected by a level of noise that is higher than for CA-LRM echoes. Indeed, the observed CA echoes are corrupted by a speckle noise resulting from the incoherent summations of $L_c = 90$ consecutive echoes for Poseidon-3 altimeter [38]. The CA-SARM results from averaging approximately 32 uncorrelated echoes (the other correlated echoes will not reduce significantly the noise level) inducing a noise increasing factor of $\sqrt{3}$ between the CA and CA-SARM echoes [39]. Fig. 10 summarizes the different steps performed to obtain the considered simulated echoes and their denominations in the rest of the paper, i.e., multi-look (or DD), delay/Doppler without migration, CA (or CA-LRM) and CA-SARM echoes.

³These echoes are known under different names: LRM-like [22], pseudo-LRM [35]–[37] or reduced-SAR (RDSAR) [36], [37]. The denomination CA-SARM has been chosen for clarity.
B. Model analysis

This section analyses the behavior of the reflected power as a function of the Doppler frequency. An example of simulation scenario corresponding to the altimetric parameters $P_u = 1$, $\text{SWH} = 0$ m and $\tau = 31$ gates is summarized in Table I. Fig. 11 shows the corresponding altimetric echoes (normalized by the maximum of the nadir echo) for different Doppler frequencies ($0, 2, 4$ and $6$ kHz). As expected, the higher power occurs at nadir, i.e., $f = 0$ Hz. This figure also shows that the echo broadens as the frequency increases which can be explained as follows. The Doppler frequency is proportional to the along-track distance (see (11)). As a consequence, the high frequencies correspond to far Doppler beams (from nadir) that intersect the large propagation circles. However, propagation circles have an increasing radius and a narrowing width for increasing time [40]. This means that the Doppler beams far from nadir intersect a lot of propagation circles (each circle correspond to a time instant) and thus the reflected echoes spread over a lot of range gates. Fig. 11 (bottom) shows example of DD echoes obtained after range migration (this figure is similar to Fig. 7 of [26]). Note that the leading edge of the multi-look echo, obtained by summing the migrated echoes, is directly affected by the high Doppler frequency echoes because of their large shape and slower leading edge. Considering
Table I
Simulation parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>13.575 GHz</td>
</tr>
<tr>
<td>Wavelength ($\lambda$)</td>
<td>2.21 cm</td>
</tr>
<tr>
<td>Bandwidth ($B$)</td>
<td>320 MHz</td>
</tr>
<tr>
<td>Altitude ($h$)</td>
<td>730 km</td>
</tr>
<tr>
<td>Burst repetition frequency (BRF)</td>
<td>85 Hz</td>
</tr>
<tr>
<td>Pulse repetition frequency (PRF)</td>
<td>18182 Hz</td>
</tr>
<tr>
<td>3 dB antenna beam width ($\theta_{3\text{dB}}$)</td>
<td>1.1388 degrees</td>
</tr>
<tr>
<td>Velocity ($v_s$)</td>
<td>7000 m/s</td>
</tr>
<tr>
<td>Pulses per burst</td>
<td>64 pulses</td>
</tr>
<tr>
<td>Burst length ($\tau_b$)</td>
<td>3.5 ms</td>
</tr>
<tr>
<td>Doppler beam width</td>
<td>327 m</td>
</tr>
</tbody>
</table>

echoes associated with different time gates (gates 31, 51, 71, 91 and 111), Fig. 12 (top) shows a decrease of the power according to Doppler frequency which is due to the weighting of the power by the Gaussian antenna gain [14]. This figure also shows a symmetrical shape of the echoes with respect to the zero Doppler frequency which is due to the absence of mispointing angle $\xi = 0^\circ$ (note that the situation can be very different in presence of mispointing as shown in [40]). These results are confirmed in Fig. 12 (bottom) which shows the Doppler spectra resulting from the summation of the powers associated with the different Doppler frequencies.

C. Importance of range migration

This section is interested in analyzing the effect of range migration on the quality of the estimated parameters. Parameter RMSEs obtained with and without range migration (with the same noise level, i.e., $L = 4$) are shown in Fig. 13 versus the sea wave height (SWH), the epoch ($\tau$) and the amplitude ($P_u$). These RMSEs have been obtained using $m = 1000$ simulated waveforms (see (30)). The parameters SWH and $\tau$ are better estimated by considering migrated DD echoes since the errors on SWH and $\tau$ are reduced by $\simeq 30$ cm and $\simeq 6$ cm respectively. However, the estimation of $P_u$ is slightly better without migration because the echo is broader and its amplitude is less sensitive to noise.
Fig. 11. Echoes for different Doppler frequencies (0, 2, 4 and 6 kHz). (top) without range migration, (bottom) with range migration. The temporal scale has been oversampled by a factor of $N_t = 16$.

Fig. 13 also shows the RMSEs when estimating CA echoes. The delay/Doppler RMSEs for $\tau$ and SWH (blue curves in Fig. 13) are lower than those obtained with CA (black curves in Fig. 13) which shows the interest of using the Doppler procedure. However, one can notice that CA provides better results for RMSE(SWH) for very small values of SWH (this result was also observed in [22]). Note finally that the obtained RMSEs are very close to the STDs (see Fig. 14) since the proposed estimator provides very small biases (see Fig. 15).
Fig. 12. (top) reflected power versus Doppler frequency for different gate numbers, (bottom) Doppler spectra after summing the powers associated with each Doppler frequency (sum of the columns of Fig. 8 (left)). The frequency scale has been oversampled by a factor of $N_f = 15$. 
Fig. 13. Parameter RMSEs for migrated and non-migrated delay/Doppler echoes and conventional echoes (1000 Monte-Carlo realizations). (a) versus SWH with $P_u = 1$ and $\tau = 31$ gates, (b) versus the epoch $\tau$ with $P_u = 1$ and SWH = 2 m, and (c) versus the amplitude $P_u$ with $\tau = 31$ gates and SWH = 2 m.
Fig. 14. Parameter STDs for migrated and non-migrated Doppler echoes and conventional echoes (1000 Monte-Carlo realizations). (a) versus SWH with $P_u = 1$ and $\tau = 31$ gates, (b) versus the epoch $\tau$ with $P_u = 1$ and SWH = 2 m, and (c) versus the amplitude $P_u$ with $\tau = 31$ gates and SWH = 2 m.
Fig. 15. Parameter Biases for migrated and non-migrated Doppler echoes and conventional echoes (1000 Monte-Carlo realizations). (a) versus SWH with $P_u = 1$ and $\tau = 31$ gates, (b) versus the epoch $\tau$ with $P_u = 1$ and SWH = 2 m, and (c) versus the amplitude $P_u$ with $\tau = 31$ gates and SWH = 2 m.
V. RESULTS FOR CRYOSAT-2 WAVEFORMS

This section is devoted to the validation of the proposed semi-analytical model for oceanic Cryosat-2 waveforms. The considered waveforms were obtained in August 2011 (the estimation was applied to the whole month of data) and were provided by the Cryosat processing prototype developed by CNES which is doing the level 1 processing and in particular the Doppler, range migration and multi-looking processings [37]. The estimated parameters of DD echoes are first compared to the results obtained with a 3 parameter estimator designed for CA-SARM echoes\(^4\). This will provide a good evaluation of the proposed delay/Doppler model. Figs. 16 shows examples of estimated Cryosat-2 echoes using the proposed model for different values of SWH. The top figures show a good fit between these two echoes especially in the leading and trailing edges of the waveform. This result is confirmed when considering the bottom figures which show the error (difference) between the two echoes. Note that the maximum difference between the real echo and its estimation is of the order of 10% which is a small value due to the presence of noise. Table II shows the averaged NREs obtained for different classes of SWH values when estimating Cryosat-2 echoes. This table shows small values of NRE which confirms the good fitting of the proposed model.

<table>
<thead>
<tr>
<th>SWH (m)</th>
<th>[0, 1]</th>
<th>[1, 2]</th>
<th>[2, 3]</th>
<th>[3, 4]</th>
<th>[4, 5]</th>
<th>[5, 6]</th>
<th>[6, 7]</th>
</tr>
</thead>
<tbody>
<tr>
<td>NRE (\times 10^{-2})</td>
<td>8.5</td>
<td>8.7</td>
<td>9.4</td>
<td>9.9</td>
<td>10.25</td>
<td>10.37</td>
<td>10.5</td>
</tr>
</tbody>
</table>

Considering the estimated parameters, the good agreement between the estimated DDA and CA-SARM parameters are illustrated in Fig. 17 which shows the estimated SWH and sea surface height anomaly (SSHA) parameters for 2 minutes of data (the SSHA is obtained by applying all environmental corrections on the estimated epoch). This agreement is confirmed in Fig. 18 showing similar correlations between the estimated SWH and SSHA using the two estimation procedures.

\(^4\)The parameters of CA-SARM echoes were estimated by the least squares procedure described in III, where \(s_k(\theta) = P(t_k)\) as defined in (1).
Table III shows the averaged differences between the estimated parameters of CA-SARM and DDA. These results are represented for SWH < 4 m since more than 90% of the processed data satisfy this constraint. The differences between the CA-SARM and DDA estimation are very low. Table III also shows the averaged STDs\(^5\) for parameters SWH and SSHA. These STDs have been obtained by considering groups of \(m = 20\) successive parameters (see (32)), i.e., one value of STD is obtained every second (the resulting STDs are known as 20 Hz STDs\(^6\)). As expected, DDA provides lower STDs than CA-SARM which is in agreement with the results of Section IV-C. Note that the equivalent CA STDs can be obtained by dividing the CA-SARM STDs by a factor of \(\sqrt{3}\) as explained previously. This provides a good evaluation of DDA when compared to CA (used in the previous altimeters such as in Poseidon-3). The STD improvement can be evaluated by computing the ratio between the CA STDs and the DDA STDs (referred to as improvement factor in Table IV). At SWH = 2 m, we obtain an SWH STD of 55 cm for CA and of 43 cm for DDA which shows an improvement factor of 1.28. Considering SSHA, we notice a CA STD of 8.16 cm and a DDA STD of 6.47 cm resulting in an improvement factor of about 1.26. Table IV compares these improvement factors with results available in the

\[^{5}\text{The averaged STDs have been obtained by averaging the obtained STDs for each 1 meter interval of SWH.}\]

\[^{6}\text{The 1 Hz STDs can be deduced from the 20 Hz STDs by dividing all results by the factor } \sqrt{20}.\]
literature. The obtained results are clearly in good agreement with those of [39], [41] (the small differences are due to the fact that it is not possible to reproduce exactly the same simulation scenario).

TABLE IV
IMPROVEMENT FACTORS OF DDA WITH RESPECT TO CA.

<table>
<thead>
<tr>
<th>Studies</th>
<th>This paper</th>
<th>[39]</th>
<th>[41]</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSHA</td>
<td>1.26</td>
<td>1.18</td>
<td>1.43</td>
</tr>
<tr>
<td>SWH</td>
<td>1.28</td>
<td>1.31</td>
<td>1.26</td>
</tr>
</tbody>
</table>

Note finally that the STD results presented in Table III are similar to those obtained in the simulation (see Fig. 13), where we have obtained better results for DDA altimetry except for small values of SWH where CA performs slightly better. The improvement factors are also in agreement with those of simulated waveforms since we have obtained 1.24 for $\tau$ and 1.19 for SWH at SWH = 2 m. These similarities between simulated and real data results validate the proposed model.
Fig. 16. Examples of estimated Cryosat-2 echoes and corresponding normalized reconstruction errors (NRE) for different values of SWH.

(a) SWH = 0.57 m and NRE = 0.07.
(b) SWH = 1.56 m and NRE = 0.065.
(c) SWH = 3.6 m and NRE = 0.12.
(d) SWH = 3.92 m and NRE = 0.101.
(e) SWH = 5.26 m and NRE = 0.112.
(f) SWH = 5.84 m and NRE = 0.102.
Fig. 17. Parameter estimates for 2 minutes of Cryosat-2 data for DD and CA-SARM echoes. (top) SWH (bottom) SSHA.

Fig. 18. Correlations between estimated SWH and SSHA parameters for CA-SARM (left) and DD (right).
VI. CONCLUSIONS

This paper defined a new semi-analytical model for delay/Doppler altimetry. A geometrical approach was used for computing an analytical expression of the flat surface impulse response. The analytical expression was obtained under the assumptions of a circular antenna pattern, no mispointing, no vertical speed effect and a uniform scattering. This analytical expression was convolved with the probability density function of the heights of the specular scatterers and the point target response of the radar leading to the mean power of a delay/Doppler altimetric waveform. A least squares approach based on the Levenberg-Marquardt algorithm was then proposed to estimate the parameters of delay/Doppler altimetric echoes. Simulation results performed on simulated data showed the good potential of delay/Doppler altimetry when compared to conventional altimetry in terms of error reduction. The analysis of real Cryosat-2 waveforms confirmed the good performance of the proposed delay/Doppler model. Extending the results obtained in this paper to a model including the mispointing angles is an interesting issue. This generalization might reduce the noise level since the Cryosat-2 echoes are known to present a mispointing of about 0.1 degree in across-track and along-track directions [42]. The consideration of the antenna ellipticity, the satellite vertical speed effect and a non constant backscattering coefficient might also improve the obtained performance. Finally, studying correlations between the estimated parameters is also an important issue that would deserve some attention.
APPENDIX A
GLOSSARY OF NOTATIONS AND NUMERICAL VALUES

Altimetric signals and parameters

- FSIR: Flat surface impulse response
- PDF: Probability density function of the heights of the specular scatterers
- PTR: Point target response
- $P$: Mean power of altimetric echo
- $G$: Power gain of the radar antenna
- $G_0$: Antenna power gain at boresight
- $\sigma_0$: Backscatter coefficient of the surface
- $\rho, \phi$: radius and angle representing the polar coordinates
- $P_u$: Amplitude of the altimetric echo
- SWH: Significant wave height
- $\tau$: Epoch
- $\xi$: Mispointing angle with respect to the $z$ axis
- $\tilde{\phi}$: Mispointing angle with respect to the $x$ axis
- $\theta$: Parameter vector
- $\sigma_s$: Parameter related to SWH
- $s(t)$: continuous multi-look signal
- $s$: discrete multi-look vector
- $y(t)$: continuous noisy multi-look signal
- $y$: discrete noisy multi-look vector (or observed signal)

Mathematical functions and notations

- $\delta$: Delta function
- $U$: Heaviside function
- $g$: Vector of residues
- $J$: Gradient vector of $g$
Numerical values

- $\lambda = 2.21$ cm: Wavelength
- $B = 320$ MHz: Bandwidth
- $h = 730$ km: Altitude
- BRF = 85 Hz: Burst repetition frequency
- PRF = 18182 Hz: Pulse repetition frequency
- $\theta_{3dB} = 1.1388$ degrees: 3 dB antenna beam width
- $\gamma = 2.85 \times 10^{-4}$: Antenna beam width parameter,
- $v_s = 7000$ m/s: Velocity
- $\alpha = 1.11$: Curvature factor
- $\tau_b = 3.5$ ms: Burst length
- $T = 3.125$ ns: Time resolution
- $F = 285$ Hz: Doppler frequency resolution
- $c = 299792458$ m/s: Speed of light
- $K = 104$: Number of samples without oversampling
- $m = 1000$: Number of simulated waveforms
- $L = 4$: Number of bursts observing a Doppler beam
APPENDIX B
AN APPROXIMATION OF THE DOPPLER BEAM FORMULA

This section is concerned with the study of the approximation (11). As explained in section II-A1, the coordinate of the along-track beam (also called along-track band) is given by (10). This formula provides an hyperbolic shape for the coordinate $y_n(t)$. This appendix discusses the approximation made in order to obtain (11) and allowing $y_n(t)$ to be time independent ($y_n$ is constant in the across-track direction). Fig. 19 shows 32 beams obtained by using (10) and (11) where (10) corresponds to hyperbolic beams and (11) to rectangular beams. This figure shows an excellent agreement between the two expressions validating the approximation $\rho(t) \ll h$. Fig. 20 shows zooms of beams #33 (central beam) and #64 (the furthest beam from nadir). Note that the difference between the hyperbolic and rectangular central beams is less than 2 cm. This negligible difference occurs at a distance of $\approx 10$ km which only affects the end of the trailing-edge of the Doppler waveform. Fig. 20 (right) finally shows the hyperbolic and rectangular beams when considering the last extreme beam #64. The difference between the two beams (about 1 m) is negligible compared to the along-track distance which is of the order of 10 km. Note finally that the approximation (11) was also proposed in [12].
Fig. 19. Rectangular and hyperbolic Doppler beams.

Fig. 20. Rectangular and hyperbolic Doppler beams for the beam #33 (left) and the beam #64 (right).
(4) and (14) introduced the analytical model when considering flat Earth surface. Some corrections have to be included in order to take into account the Earth curvature. Fig. 21.a and 21.b show the geometry of the scene in the two cases. Introducing Earth curvature is obtained by changing the expression of \( r \) in (2). In the case of a round Earth, \( \rho \) and \( l_r \) are related according to the following expression

\[
\rho = R \arcsin \left( \frac{l_r}{R} \right) \tag{34}
\]

where \( R = 6378137 \) m is the Earth radius, \( \arcsin(.) \) is the inverse sine function and \( l_r \) is the distance between the illuminated point and the line linking the satellite to the center of Earth (see Fig. 21.b). This equation shows that \( \rho \approx l_r \) for small angle geometry. Indeed, by considering a pessimistic case corresponding to the large value of \( l_r = 10 \) km, we obtain \( \rho - l_r = 4 \) mm which is a negligible difference. Therefore, we will consider \( \rho = l_r \) in the rest of this section.

The distance \( r \) between the satellite and the observed surface is given by the following formula

\[
r = \sqrt{\left( R \sin \beta \right)^2 + \left( h + R - R \cos \beta \right)^2} \tag{35}
\]
where $\beta$ is the angle between the illuminated point and the line linking the satellite to the center of Earth (see Fig. 21.b). Considering that $R \sin \beta = \rho$ and $\rho^2 \ll R^2$, (35) reduces to

$$r \simeq \sqrt{h^2 + \alpha \rho^2}$$

(36)

where $\alpha = 1 + \frac{h}{R} = 1.11$ is the curvature factor. By replacing (36) in (2), straightforward computations show that $\epsilon(t') = \frac{1}{\sqrt{\alpha}} \sqrt{\left(\frac{\epsilon_0}{2h}\right)^2 - 1}$. When using the two-way incremental ranging time $t = t' - \frac{2h}{c}$ and the approximation $\frac{ct}{h} \ll 1$ (valid for spaceborne altimetry [1]), we can see that $\epsilon(t) \simeq \sqrt{\frac{ct}{h}}$ used for a flat Earth has to be replaced by $\epsilon(t) \simeq \sqrt{\frac{ct}{\alpha h}}$ for a round Earth. In other words, to move from a flat Earth to a round Earth, it is sufficient to divide $t$ by $\alpha$. Note that the same change of variable was proposed in [6] for conventional altimetry. Note finally the negligible effect of Earth curvature on Doppler band (see Appendix D for more details).
APPENDIX D
EFFECT OF EARTH CURVATURE ON DOPPLER BEAM

This section is interested in the analysis of Earth curvature effects on the Doppler beams. In order to obtain the Doppler beam formula for round Earth, we have first to generalize the expression of \( r_n(t) \) given in (9) as follows

\[
r_n(t) = \sqrt{h^2 + \alpha \rho^2(t)}
\]

which leads to the following width of the along-track beam

\[
y_n(t) = \left( \frac{\lambda f_n}{2v_s} \right) \sqrt{h^2 + \alpha \rho^2(t)}.
\]

Note that we always have \( \alpha \rho^2(t) \ll h^2 \) which means that the effect of Earth curvature (represented in (38) by the coefficient \( \alpha \)) is negligible since we can simplify (38) to (11) as explained in Section II-B1.

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REFERENCES


C. Gommenginger, P. Cipollini, D. Cotton, S. Dinardo, and J. Benveniste, “Finer, better, closer: Advanced capabilities of


